KINETIC EQUATION FOR AN OSCILLATOR IN A RANDOM EXTERNAL FIELD

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An analysis is given of the behavior of an oscillator in an external random field, assuming that the time of interaction is much longer than the period of the oscillator. The kinetic equation describing the coordinate and velocity distribution as functions of time is derived.

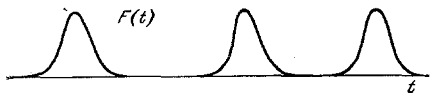

Fig. 1

The problem is analogous to the motion of a charged particle in an external magnetic field in the presence of collisions.

A well-known situation in the theory of interaction of plasma particles with the wave field due to instability is the possibility of considering the motion of an individual particle as if it were taking place in a given random field. If in addition there is an external magnetic field, the motion of the particle is analogous to the motion of an oscillator in a random force field. Random forces acting on the oscillator can be imagined as a sequence of pulses with a known random distribution of pulse heights and intervals between pulses. Depending on the pulse length τ (the time of "collision") and the period ω of the oscillator, there are two possible limiting cases, namely: 1) $\omega\tau \ll 1$ and 2) $\omega\tau \gg\gg 1$. The first case has been investigated in sufficient detail. In this paper we consider the second case in the approximation where the mean interval between pulses is much greater than τ (infrequent "collisions").

The basic equation describing the motion of the oscillator will be

$$x^{\cdot\cdot} + [\omega^2 + F(t)]\, x = 0\,, \qquad (1)$$

where F(t) is a sequence of sufficiently smooth pulses (the effect of a pulse on a particle will be referred to as a collision) with a known random distribution of pulse shapes and intervals between pulse (Fig. 1).

The condition

$$\omega\tau \gg 1 \qquad (2)$$

enables us to write out the solution of Eq. (1) in the WKB approximation [1]

$$x = Ax_+ + Bx_-,$$

$$x_\pm = \Omega^{-1/2} \exp\left\{\pm i \int^t \Omega(t')\,dt'\right\}, \quad \Omega = \sqrt{\omega^2 + F(t)}. \qquad (3)$$

In the interval between collisions $\Omega \approx \omega$, and the solution given by Eq. (3) represents ordinary oscillations. A collision leads to a change in the adiabatic invariant (E is the energy of the oscillator):

$$I = E/\Omega\,. \qquad (4)$$

This is analogous to reflection from a barrier in quantum mechanics. Consider the complex t plane. The asymptotic solutions Eq. (3) are of the form shown in Fig. 2. It is assumed that the nearest singular points to the real axis are t_k, $\bar{t}_k$ (bars represent complex conjugates) at which $\Omega_2(t)$ has a simple zero

$$\Omega^2(t) = \psi(t) \prod_k (t - t_k)(t - \bar{t}_k)(5), \qquad (5)$$

where $\psi(t)$ has singularities or zeros at points with imaginary parts much greater than Im t_k. The change in the adiabatic invariant subject to the condition given by Eq. (5) occurs when a sufficiently narrow region near the points O_k is intersected (Fig. 2). It was calculated in [2].

Let us introduce the translation operator T_n^+

$$x(t + T_n) = T_n^+ x(t)\,, \qquad (6)$$

where T_n is the interval between O_{n+1} and O_n, and the time t lies in the interval (O_{n-1}, O_n). We shall take the operator T_n^+ in the form given in [3]:

$$T_n^+ = \begin{pmatrix} \sqrt{1 + e^{-2\delta_n}}\, e^{i(1/2\pi + S_n + \varphi_n)} & e^{-\delta_n - i(1/2\pi - S_n)} \\ e^{-\delta_n + i(1/2\pi - S_n)} & \sqrt{1 + e^{-2\delta_n}}\, e^{-i(1/2\pi + S_n + \varphi_n)} \end{pmatrix}, \qquad (7)$$

$$\delta_n = -i \int_{\bar{t}_n}^{t_n} \Omega(t')\,dt' > 0, \qquad S_n = \int_{t_n}^{t_{n+1}} \Omega(t')\,dt' > 0\,. \qquad (8)$$

The operator T_n^+ will then act on the column vector with components $A_n x_+$, $B_n x_-$. The order of magnitude on the phase φ_n in Eq. (7) is not greater than unity, and its exact form will not be necessary below. The operator T_n^+ is given by

$$T_n^+ = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1\,. \qquad (9)$$

It will be convenient to take x in a real form, i. e., $B = \bar{A}$. It follows from Eq. (8) that if $B_n = \bar{A}_n$ then $B_{n+1} = \bar{A}_{n+1}$. Hence, according to Eq. (4),

$$I_n = |A_n|^2\,. \qquad (10)$$

Let us now take into account the fact that the collisions are infrequent. We have

$$S_n \approx \omega T_n \gg 1\,, \qquad (11)$$

and the translation operator assumes the form

$$T_n^+ = \begin{pmatrix} \sqrt{1 + \varepsilon_n^2}\, e^{i\omega T_n} & \varepsilon_n e^{i\omega T_n} \\ \varepsilon_n e^{-i\omega T_n} & \sqrt{1 + \varepsilon_n^2}\, e^{-i\omega T_n} \end{pmatrix},$$

$$\varepsilon_n = \frac{1}{2} e^{-\delta_n}\,. \qquad (12)$$

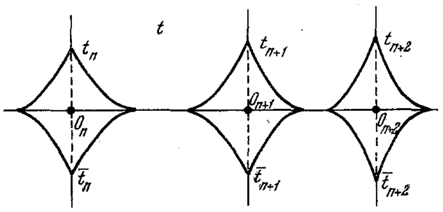

Fig. 2

Consider the equation

$$\xi^{\cdot\cdot} + \left[\omega^2 + \sum_k u_k \delta(t - t_k^*)\right] \xi = 0\,, \qquad (13)$$

whose solution in the interval (t_{n-1}^*, t_n^*) will be written in the form

$$\xi(t) = A_n^* \xi_+ + \bar{A}_n^* \xi_-, \qquad \xi_\pm = e^{\pm i\omega t}. \qquad (14)$$

The translation operator T_n^+ will be defined by analogy with Eq. (6):

$$\xi(t + T_n^*) = T_n^{+*} \xi(t) \qquad (T_n^* = t_{n+1}^* - t_n^*)\,. \qquad (15)$$

We then have

$$T_n^{+*} =$$

$$= \begin{pmatrix} \sqrt{1 + {}^1/_4 u_n^2/\omega^2}\, e^{i(\varphi_n^* + \omega T_n^*} & \frac{u_n}{2\omega} e^{i({}^1/_2\pi + \omega T_n^*)} \\ \frac{u_n}{2\omega} e^{-i({}^1/_2\pi + \omega T_n^*)} \sqrt{1 + {}^1/_4 u_n^2/\omega^2} & e^{-i(\varphi_n^* + \omega T_n^*)} \end{pmatrix},$$

$$\varphi_n^* = \operatorname{arc\,tg}(u_n/2\omega).$$

If we compare Eqs. (12) and (16) and take Eq. (11) into account, we see that the initial equation given by Eq. (1) can be replaced by Eqs. (12) if we make the substitutions

$$\xi = \sqrt{\Omega}\, x, \quad \varepsilon_n = u_n/2\omega, \quad T_n^* = T_n. \tag{17}$$

Let us now derive the kinetic equation. Let $f(\xi, \eta, t)$ be the probability density that at a time t the variable ξ lies in the range $(\xi, \xi + d\xi)$, and the variable $\eta = \xi^{\cdot}$ lies in the range $(\eta, \eta + d\eta)$, where

$$\int f(\xi, \eta, t)\, d\xi\, d\eta = 1.$$

The points t_k^* in Eq. (13) will be assumed to be distributed in accordance with Poisson's law, i.e., the probability that the interval between t_k^* and t_{k+1}^* lies in the range $(T, T + dT)$ is given by

$$P(T)\, dT = \lambda\, e^{-\lambda T}\, dT. \tag{18}$$

We will also assume that the probability density $w(\varepsilon)$ that ε_n (for any n) lies in the range $(\varepsilon, \varepsilon + d\varepsilon)$ is known. The equation for $f(\xi, \eta, t)$ in the case of Eq. (13) was discussed in [4]. We shall give it without proof:

$$\frac{\partial f(\xi, \eta, t)}{\partial t} = \left(\omega^2 \xi \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial \xi}\right) f(\xi, \eta, t) +$$

$$+ \lambda \int d\varepsilon' w(\varepsilon') f(\xi, \eta + \omega\varepsilon'\xi, t) - \lambda f(\xi, \eta, t). \tag{19}$$

In particular, if

$$w(\varepsilon') = 1/\pi\, \delta(\varepsilon' - \varepsilon),$$

we have

$$\frac{\partial f(\xi, \eta, t)}{\partial t} = \left(\omega^2 \xi \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial \xi}\right) f(\xi, \eta, t) +$$

$$+ \lambda\, [f(\xi, \eta + \omega\varepsilon\xi, t) - f(\xi, \eta, t)] \tag{20}$$

For large δ, i.e., very small ε, we have from Eq. (20) the Fokker-Planck equation

$$\frac{\partial f}{\partial t} \approx (\omega^2 + \varepsilon\lambda\omega)\, \xi \frac{\partial f}{\partial \eta} - \eta \frac{\partial f}{\partial \xi} + \lambda \frac{\omega^2\varepsilon^2}{2} \xi^2 \frac{\partial^2 f}{\partial \eta^2}, \tag{21}$$

with the coefficient of diffusion given by

$$D \sim {}^1/_2 \lambda \omega^2 \varepsilon^2 \xi^2. \tag{22}$$

According to Eq. (12), the quantity ε is exponentially small and, consequently, the coefficient of diffusion is also small. We shall see below that the second moments of f which, in view of Eqs. (10) and (17), are proportional to the adiabatic invariant I for the oscillator, increase with time. This ensures that Eq. (21) eventually becomes invalid. The condition for the validity of this equation is

$$\xi \ll \xi_0 = \frac{2}{\omega\varepsilon} \left[\frac{\partial}{\partial \eta} \ln f\right]^{-1}. \tag{23}$$

Let us now introduce the second moments of the distribution function f: $<\xi^2>$, $<\xi\eta>$, $<\eta_2>$, where the angle brackets represent averaging over the distribution $f(\xi, \eta, t)$. The equations for these moments are given in [4] and are readily derived from Eq. (20). They are given by

$$\frac{d}{dt} \langle \xi^2 \rangle = 2 \langle \xi\eta \rangle,$$

$$\frac{d}{dt} \langle \xi\eta \rangle = -(\omega^2 + \lambda\varepsilon\omega) \langle \varepsilon^2 \rangle + \langle \eta^2 \rangle,$$

$$\frac{d}{dt} \langle \eta^2 \rangle = \lambda\varepsilon^2\omega^2 \langle \xi^2 \rangle - 2(\omega^2 + \lambda\varepsilon\omega) \langle \varepsilon\eta \rangle. \tag{24}$$

The solution of Eq. (24) is proportional to $e^{\Gamma t}$, where Γ represents the roots of the equation

$$\Gamma^3 + 4(\omega^2 + \lambda\varepsilon\omega)\, \Gamma - 2\lambda\varepsilon^2\omega^2 = 0. \tag{25}$$

The last equation always has one unstable root $\Gamma_0 > 0$, and in our case

$$\Gamma_0 \approx {}^1/_2\, \varepsilon^2\lambda = {}^1/_8\, \lambda e^{-2\delta}. \tag{26}$$

Therefore, the characteristic time for the development of instability is sufficiently long:

$$\tau_R \sim 8 e^{2\delta} \lambda^{-1}. \tag{27}$$

In accordance with Eq. (19), all the second moments are proportional to I, and their average increase with time means that the quantity $<I> = <\xi^2 + \eta^2/\Omega^2>$ will also increase.

It is occasionally convenient to rewrite Eq. (19) in terms of action-phase variables. Let

$$I = \sqrt{\xi^2 + \omega^{-2}\eta^2}, \qquad z = \eta/\omega\xi. \tag{28}$$

Equation (19) then assumes the form

$$\frac{\partial F(I, z)}{\partial t} = \omega \frac{\partial}{\partial z} [(1 + z^2) F(I, z)] +$$

$$+ \lambda \left\{ F(I^*, z^*) \frac{\partial(I^*, z^*)}{\partial(I, z)} - F(I, z) \right\}, \tag{29}$$

where

$$I^* = I \left(\frac{1 + (z + \varepsilon/\omega)^2}{1 + z^2} \right)^{1/2}, \quad z^* = z + \frac{\varepsilon}{\omega},$$

$$F(I, z)\, dI dz = f(\xi, \eta)\, d\xi d\eta. \tag{30}$$

If we integrate Eq. (29) with respect to I we obtain the kinetic equation for the phases (more precisely, for the cotan of the phase z). However, the kinetic equation for

$$F(I) = \int_{-\infty}^{\infty} F(I, z)\, dz$$

alone cannot be obtained.

The above method of deriving the kinetic equation can also be applied to more complicated systems. The method requires a knowledge of the translation operator T^+ in the WKB approximation.

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REFERENCES

1. J. Heading, Introduction to Phase-Integral Methods [Russian translation], Izd. Mir, 1965.

2. A. Dykhne, "Quantum transitions in the adiabatic approximation," Zh. eksperim. i teor. fiz., vol. 38, no. 2, 1960.

3. G. M. Zaslavskii, Lectures on the Applications of the WKB Method in Physics [in Russian], Izd. Novosibirsk. un-ta, 1965.

4. H. Z. Frisch and S. P. Lloyd, "Electron levels in a one-dimensional random lattice," Phys. Rev., vol. 120, no. 4, 1960.

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